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Time evolution in general gauge theories[1]

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Introduction

In this talk I will discuss some properties of time evolutions in general gauge theories within a BRST quantization [1]. More precisely I will discuss the choices of Hamiltonians within the Hamiltonian framework set up by Batalin, Fradkin and Vilkovisky which is called the BFV formulation [2] (for a review see *e.g.* [3]). This I will do from the point of view of an operator formulation for inner product solutions within the BFV scheme which I have been developing during some years [4]-[8]. This formalism turns out to yield more information about quantum properties than just an effective BRST invariant Lagrangian or Hamiltonian formulation. In fact, an effective Hamiltonian is more difficult to extract within this scheme, but the procedure provides for a deeper understanding of the standard BFV prescriptions. These results will be briefly reviewed. As a particular example of a natural consequence I will at the end show that QED is coBRST invariant. However, let me first review the standard BFV formulation.

Standard BFV-BRST

Within the BFV formulation Hamiltonians of general gauge theories are assumed to have the form

$$H = H_0 + \int v_i\theta_i, \tag{1}$$

where v_i are Lagrange multipliers and θ_i constraint variables. (Repeated indices are summed over and integrals are over space coordinates.) H_0 and θ_i satisfy the super Poisson bracket conditions

$$\{H_0, \theta_i\} = C_{ij}\theta_j, \quad \{\theta_i, \theta_j\} = C_{ijk}\theta_k. \tag{2}$$

where C_{ij} and C_{ijk} may be functions on the phase space. In the corresponding BRST quantization BFV introduces the following additional degrees of freedom:

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- π_i – conjugate momenta to the Lagrange multipliers v_i. (They are additional abelian constraint variables.)
- $\mathcal{C}_i$, $\mathcal{P}_i$ – ghosts and their conjugate momenta.
- $\bar{\mathcal{C}}_i$, $\bar{\mathcal{P}}_i$ – antighosts and their conjugate momenta.

Their Grassmann parities and ghost numbers are

$$\varepsilon(\mathcal{C}_i) = \varepsilon(\bar{\mathcal{C}}_i) = \varepsilon(\theta_i) + 1, \quad gh(\mathcal{C}_i) = 1 = -gh(\bar{\mathcal{C}}_i). \tag{3}$$

In this extended phase space the BRST charge is given by

$$Q = \int \left(\mathcal{C}_i\theta_i + \ldots + \bar{\mathcal{P}}_i\pi_i\right), \tag{4}$$

where the dots indicates terms determined by the super Poisson bracket condition $\{Q, Q\} = 0$. The Hamiltonian for the effective BRST invariant theory is defined to be

$$H_{BRST} = H_0' + i\{Q, \psi\}, \quad \{H_{BRST}, Q\} = 0, \tag{5}$$

where

$$H_0' = H_0 + \text{ghost dependent terms}, \quad \{H_0', Q\} = 0, \tag{6}$$

and where in turn ψ is an odd gauge fixing fermion which usually is chosen such that

$$H_{BRST} = H + \int (\ )\pi_i + \text{ghost dependent terms}. \tag{7}$$

Such a Hamiltonian leads to a BRST invariant effective Lagrangian of the standard form

$$\mathcal{L}_{BRST} = \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh}. \tag{8}$$

The general allowed form for ψ as prescribed by BFV is

$$\psi = \int \left(\mathcal{P}_i v_i + \bar{\mathcal{C}}_i\chi_i\right), \tag{9}$$

where χ_i are gauge fixing variables to θ_i. (The matrix $\{\chi_i, \theta_j\}$ is required to be invertible.)

One may observe that neither H_0' nor ψ are uniquely determined. For instance, in an abelian gauge theory H_{BRST} is invariant under the transformations

$$H_0 \longrightarrow H_0 + x_i\theta_i, \quad \psi \longrightarrow \psi + \mathcal{P}_i x_i \tag{10}$$

for any BRST invariant variable x_i.

Operator quantization on inner product spaces.

Case 1: $H_0' = 0$

This case includes all reparametrization invariant theories, such as particles, strings, and gravity. The operator quantization proceeds here as follows: Quantize all degrees of freedom and construct an extended inner product state space V. Physics is then what is contained in the subspace $V_{ph} \subset V$ defined by $QV_{ph} = 0$. V_{ph} is degenerate since the zero norm states QV is contained in V_{ph}. The nondegenerate inner product space is therefore $V_s = V_{ph}/QV$, the states of BRST singlets. V_s is an inner product space if V is an inner product space. An important concept in this connection is the coBRST charge ${}^*Q$ [9]. It is defined by

$$ {}^*Q = \eta Q \eta, \tag{11} $$

where η is an hermitian metric operator such that $\eta^2 = 1$ and $\langle u|\eta|u\rangle \geq 0 \; \forall |u\rangle \in V$. Thus, η maps V onto a Hilbert space and ${}^*Q$ is just the hermitian conjugate of Q in this Hilbert space. We have ${}^*Q^2 = 0$. In terms of the coBRST charge the BRST singlets $|s\rangle \in V_s$ are determined by

$$ Q|s\rangle = {}^*Q|s\rangle = 0 \tag{12} $$

or equivalently

$$ \triangle|s\rangle = 0, \quad \triangle \equiv [Q, {}^*Q]_+. \tag{13} $$

Now it is usually very difficult to find the appropriate inner product space V. (There are even cases which allow for several different choices.) Fortunately, there is a possibility to construct formal operator expressions for the singlets $|s\rangle$ without prescribing V. In fact, such expressions will at the end tell you the appropriate prescription for V [8]. Since this formalism is not yet completely rigorously proved, I will present the main ingredients as a set of proposals:

> *Proposal 1*: If $Q = \delta + \delta^\dagger$, where δ, $\delta^\dagger$ are independent nilpotent operators each containing effectively half the constraints of Q, then the solutions of $\delta|ph\rangle = \delta^\dagger|ph\rangle = 0$ are formally inner product solutions what concerns the unphysical degrees of freedom.

> *Proposal 2*: Q for any gauge theory in BFV form may be decomposed as $Q = \delta + \delta^\dagger$, where δ, $\delta^\dagger$ are independent and each containing effectively half the constraints of Q and such that $\delta^2 = 0$ and $[\delta, \delta^\dagger]_+ = 0$.

> *Proposal 3*: The formal solutions of $\delta|ph\rangle = \delta^\dagger|ph\rangle = 0$ have up to zero norm states the general form $|ph\rangle = e^{[Q,\psi]_+}|\phi\rangle$ where ψ is an odd gauge fixing fermion of the form (9), and where $|\phi\rangle$ satisfies simple hermitian conditions.

Proposal 1 has been shown to be valid in all investigated cases. Proposal 2 has been proved for general Lie group theories [4]. Concerning proposal 3 the following may be said: Formal inner product solutions of the form $|ph\rangle = e^{[Q,\psi]_+}|\phi\rangle$ exist for any gauge theory if Q is

in BFV form. In fact, in [6] it was shown that the BRST singlets $|s\rangle$ locally may be written as $|s\rangle = e^{[Q,\psi]_+}|\phi_s\rangle$ where $|\phi_s\rangle$ is a ghost and gauge fixed $|\phi\rangle$. There it was shown that $D_i|s\rangle = 0$ where D_i are a complete set of BRST doublets (C,$[Q,C]_\pm$-pairs) satisfying $[D_i, D_j]_\pm = c_{ijk}D_k$, and that $[D_i, D_j^\dagger]_\pm$ is an invertible matrix operator. ($\langle\phi|\phi\rangle$ is undefined while $\langle\phi|e^{[Q,\psi]_+}|\phi\rangle$ is well defined and independent of ψ.)

What is the form of the coBRST charge within the BFV formalism? It turns out that the coBRST charge has the form of an allowed gauge fixing fermion [7]. However, one may notice that $|s\rangle = e^{[Q,{}^*Q]_+}|\phi_s\rangle = e^{\triangle}|\phi_s\rangle$ does not satisfy ${}^*Q|s\rangle = 0$. Only $|s'\rangle = U|s\rangle$ does, where U is a unitary operator[7].

Case 2: $H_0' \neq 0$

This case may always be transformed to case 1 ($H_0' = 0$) by making the gauge theory reparametrization invariant. The transformed theory may then be treated as before. This is the fundamental approach here. *Thus, nontrivial time evolution does not cause any basic problems.* However, problems do occur whenever one tries to find inner product solutions without going to the corresponding reparametrization invariant theory. A simple and natural prescription for inner product solutions in the latter case is first to construct inner product solutions as in case 1 ignoring H_0' and then to require $|ph, t\rangle$ or $|s, t\rangle$ to be determined by a Schrödinger equation with the Hamiltonian H_0'. From the corresponding reparametrization invariant theory one finds that this is possible provided the gauge fixing conditions satisfy some weak conditions like

$$[Q, [H_0', \psi]_-]_+ = 0, \quad [\psi, [H_0', \psi]_-]_+ = 0. \tag{14}$$

That the Hamiltonian must be H_0' follows from the fact that the BRST charge in the corresponding reparametrization invariant theory is

$$\tilde{Q} = Q + \mathcal{C}(\pi + H_0') + \bar{\mathcal{P}}\pi_v, \tag{15}$$

where $\mathcal{C}$ and $\bar{\mathcal{P}}$ are new ghost variables, π_v is the conjugate momentum to a new Lagrange multiplier, and π is the conjugate momentum to a dynamical time variable. Thus, since $\pi + H_0'$ is a new constraint variable it is easily understood that $(\pi + H_0')|\ \rangle = 0$ is a natural equation, and this is the Schrödinger equation with H_0' as Hamiltonian. It turns out that the BRST singlets satisfy this Schrödinger equation strictly under weak conditions like (14). However, the problem with this procedure is that ψ and H_0' are not uniquely given. It is well known that one by unitary transformations may change the constraint variables in the BRST charge and H_0' is part of a constraint in the reparametrization invariant theory (15). On the other hand, this freedom may be used to find a H_0' and ψ's satisfying conditions (14). Whether or not this is possible in general is unclear though.

According to the procedure above we have in the case when H_0' has no explicit time dependence

$$|ph, t\rangle = e^{-iH_0't}|ph\rangle. \tag{16}$$

Since the first condition in (14) implies $[H_0', [Q, \psi]_+]_- = 0$ by means of the Jacobi identities, (16) combined with proposal 3 implies

$$|ph, t\rangle = e^{-iH_0' t + [Q,\psi]_+}|\phi\rangle. \tag{17}$$

This leads to

$$\langle ph, t'|ph, t\rangle = \int d\omega' d\omega\, \phi'^*(\omega'^*)\phi(\omega)\langle\omega', t'|\omega^*, t\rangle, \tag{18}$$

where ω denotes all coordinates of the original BRST invariant theory. (Due to the indefinite metric state space not all hermitian operators have real eigenvalues.) After the replacement $\psi \rightarrow (t' - t)\psi/2$, which is possible for any finite $t' - t \neq 0$, one may derive the path integral representation [5, 1]:

$$\langle\omega', t'|\omega^*, t\rangle = \int D\omega D\pi_\omega\, \exp\left\{ i \int_t^{t'} (\pi_\omega \dot{\omega} - H_{BRST}) \right\}, \tag{19}$$

where

$$H_{BRST} = H_0' + i\{Q, \psi\} \tag{20}$$

is the effective Hamiltonian function in agreement with the BFV prescription (5). Notice, however, that H_{BRST} is not real in general. Often a real effective Hamiltonian requires us to choose an imaginary time t [5].

One may notice that the second condition in (14) is satisfied if $\psi^2 = 0$. If $[H_0', \psi]_- = 0$ and $\psi^2 = 0$ then the effective Hamiltonian H_{BRST} in the path integral (19) satisfies $\{H_{BRST}, \psi\} = 0$ which implies that ψ generates a new nilpotent symmetry. That this may be realized is shown in the following example.

Example: QED

Consider for simplicity the Lagrangian density for a free electromagnetic field (the metric is time-like)

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{21}$$

The canonical momenta to A_μ are

$$E^\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = F^{\mu 0}. \tag{22}$$

$E^0 = 0$ is a primary constraint. The Hamiltonian density, which is equal to the canonical energy density T^{00}, is given by

$$\mathcal{H} = -\frac{1}{2}E^i E_i + E^i \partial_i A^0 + \frac{1}{4}F^{ij}F_{ij}. \tag{23}$$

The Hamiltonian equations of motion are generated by

$$H \equiv \int d^3x \left(\mathcal{H}(x) + \dot{A}^0 E^0\right), \tag{24}$$

where $\dot{A}^0$ is an arbitrary function which represents the gauge freedom. (The Lorentz condition $\partial_\mu A^\mu = 0$ demands $\dot{A}^0 = -\partial_i A^i$.) Since

$$\dot{E}^0(x) = \{E^0(x), H\} = \partial_i E^i(x) \tag{25}$$

consistency requires the secondary constraint (Gauss' law) $\partial_i E^i = 0$.

The standard Faddeev-Popov Lagrangian for QED is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 - i\partial_\mu \bar{\mathcal{C}}\partial^\mu \mathcal{C}, \tag{26}$$

where α is a real parameter. The corresponding Hamiltonian within the BFV scheme is given by (20) where the standard choice is

$$\mathcal{H}_0 = -\frac{1}{2}E^i E_i + \frac{1}{4}F^{ij}F_{ij}, \quad Q = \int d^3x(\mathcal{C}\partial_i E^i - \bar{\mathcal{P}}E^0). \tag{27}$$

(We have a minus sign in Q since $\pi_v = -E^0$.) The gauge fixing fermion is

$$\psi = \int d^3x(\mathcal{P}v + \bar{\mathcal{C}}\chi), \tag{28}$$

where $v = -A^0$ and $\chi = \partial_i A^i + E^0/2\alpha$. However, this ψ is neither conserved nor nilpotent, and it does not satisfy the conditions (14).

Now there is another option for $\mathcal{H}_0$, namely

$$\mathcal{H}_0 = -\frac{1}{2}E^i E_i + \frac{1}{2\nabla^2}(\partial_i E^i)^2 + \frac{1}{4}F^{ij}F_{ij}, \quad \nabla^2 \equiv -\partial_i \partial^i. \tag{29}$$

This choice determines the Lagrange multiplier v to be

$$v \equiv -A^0 - \frac{1}{2\nabla^2}\partial_i E^i. \tag{30}$$

Exactly the same effective Hamiltonian as before is now obtained by the formula (20) with $\mathcal{H}_0$ given by (29) and ψ given by (28) now with v as in (30) and $\chi = \partial_i A^i + E^0/2\alpha$. In distinction to the previous construction $\mathcal{H}_0$ in (29) satisfies the strong condition $\{\psi, H_0\} = 0$. For $\alpha = 1$ (the Feynman gauge) ψ is furthermore nilpotent and may be identified with a coBRST charge. In this case we have $\{\psi, H_{BRST}\} = 0$ and ψ generates a symmetry transformation. It is

$$rA^0 = \frac{1}{2}\bar{\mathcal{C}}, \quad rA^i = -\frac{1}{2\nabla^2}\partial^i \dot{\bar{\mathcal{C}}}, \quad r\mathcal{C} = \frac{1}{2}i(A^0 - \frac{1}{\nabla^2}\partial_i \dot{A}^i), \quad r\bar{\mathcal{C}} = 0. \tag{31}$$

(Like the BRST transformation it is only nilpotent on-shell.) Of course, also the bosonic charge, $i\{Q,\psi\}$, is conserved and generates a symmetry transformation. The symmetry transformation (31) was also given in [10]. (I am thankful to Joaquim Gomis for pointing out this reference to me.)

The corresponding construction for Yang-Mills theories is more difficult. The standard BRST fixed Lagrangian does not allow for a conserved coBRST charge due to the Gribov ambiguities [11]. (χ contains the Coulomb gauge $\partial_i A^i_a$.)

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